Rate-Distortion Analysis for Sampled Correlated Cyclostationary Gaussian Processes

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April 15th, 2026

ECE Departmental Seminar

Outline

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- Man-made signals are typically generated via **repetitive** operations, resulting in **cyclostationary** statistics.
  - A **wide-sense cyclostationary (WSCS)** process has a *periodic mean* and a *periodic autocorrelation function (AF)* with the *same* period:
  - A real **continuous-time (CT)** random process $X(t)$, $t \in \mathcal{R}$, is called **WSCS** if $\exists \mathbf{T_c} \in \mathcal{R}^{++}$, s.t. $\forall\ \lambda \in \mathcal{R}$,

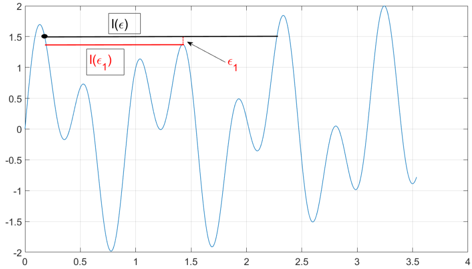
$$m_X(t) \triangleq \mathbb{E}\{X(t)\} = m_X(t + \mathbf{T_c}),$$
$$c_X(t, \lambda) \triangleq \mathbb{E}\{X(t) \cdot X(t + \lambda)\} = c_X(t + \mathbf{T_c}, \lambda).$$

  - Similarly, we define a **discrete-time (DT) WSCS** process.

- For example, *baseband OFDM signals* are **CT WSCS**.

- A real DT deterministic function $f[i]$, $i \in \mathcal{Z}$, is called **almost periodic**, if
  - $\forall \epsilon \in \mathcal{R}^{++}$, $\exists l_\epsilon \in \mathcal{N}^{+}$, which satisfies:
  - $\forall \alpha \in \mathcal{Z}$, $\exists \Delta \in [\alpha, \alpha + l_\epsilon)$, s.t.

$$\sup_{i \in \mathcal{Z}} |f[i+\Delta] - f[i]| < \epsilon.$$



- A real **zero-mean** DT random process $X[i]$, $i \in \mathcal{Z}$, is called **wide-sense almost cyclostationary (WSACS)** if $c_X[i, \Delta]$ is *almost periodic* in $i \in \mathcal{Z}$.

Sampling a CT WSCS process

- Given a **CT WSCS source process** $X_c(t)$ with the **period of statistics** of T_c.

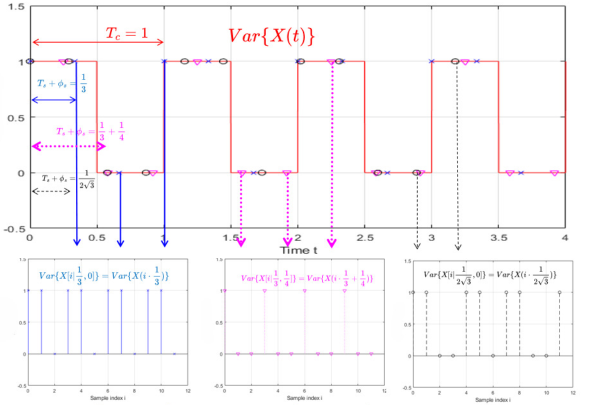
- Sample $X_c(t)$ using the **interval** $T_s(\epsilon) \triangleq \frac{T_c}{p+\epsilon}$, where $p \in \mathcal{N}^+$ and $\epsilon \in (0, 1)$, and the **initial sampling phase** $\phi_s \in [0, T_c)$

$$\longrightarrow \text{DT sampled process:}$$

$$X_\epsilon^{\phi_s}[i] \triangleq X_c\big(T_s(\epsilon) \cdot i + \phi_s\big).$$

  - **Synchronous sampling**: $\epsilon \in \mathcal{Q} \longrightarrow X_\epsilon^{\phi_s}[i]$ is a **DT WSCS process**;
  - **Asynchronous sampling**: $\epsilon \notin \mathcal{Q} \longrightarrow X_\epsilon^{\phi_s}[i]$ is a **DT WSACS process**.

Sampling a CT WSCS process



Scenario 1: synchronous sampling, zero initial sampling phase

Scenario 2: synchronous sampling, non-zero initial sampling phase

Scenario 3: asynchronous sampling, zero initial sampling phase

Sampling a **CT WSCS** process vs. Sampling a **CT stationary** process

NOT *necessarily* a **DT WSCS** process. | *Necessarily* a **DT stationary** process.

- Denote with $\mathcal{X}$ the **source alphabet** and with $\hat{\mathcal{X}}$ the **reconstruction alphabet**.

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- A **lossy source code** (M, l) consists of
  - a **message set** of **size** M;
  - an **encoder**: $f_l(\cdot) : \mathcal{X}^l \mapsto \{0, 1, 2, \ldots, M-1\}$;
  - a **decoder**: $g_l(\cdot) : \{0, 1, 2, \ldots, M-1\} \mapsto \hat{\mathcal{X}}^l$.

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- The **squared-error distortion function** between a **source sequence** $\{x_i\}_{i=0}^{l-1} \equiv x^l$ and a **reconstruction sequence** $\{\hat{x}_i\}_{i=0}^{l-1} \equiv \hat{x}^l$ is given as

$$d_{se}\left(x^l, \hat{x}^l\right) \triangleq \frac{1}{l} \sum_{i=0}^{l-1} (x_i - \hat{x}_i)^2 \equiv \overline{d}\left(x^l, \hat{x}^l\right).$$

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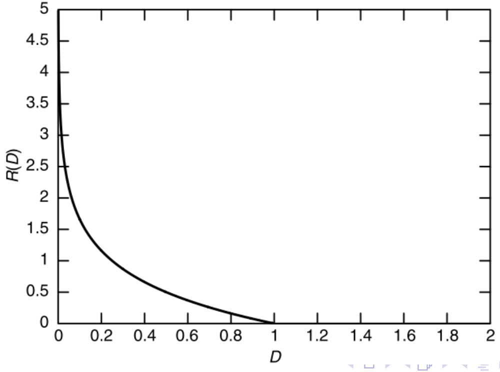
- A **rate-distortion pair** (R, D) is **achievable**, if there exists a **lossy source code** $(2^{lR}, l)$ satisfying

$$\limsup_{l \to \infty} \mathbb{E}\left\{\overline{d}\left(\mathbf{X}^l, g_l\left(f_l(\mathbf{X}^l)\right)\right)\right\} \leq D.$$

- The **rate-distortion function (RDF)** is the **infimum** of all **code rates** R, for which the **rate-distortion pair** (R, D) is **achievable**.
  - **RDF** for an **i.i.d. Gaussian source** $\mathcal{N}(0, \sigma^2)$:

$$R(D) = \begin{cases} \frac{1}{2} \log \frac{\sigma^2}{D}, & 0 \leq D \leq \sigma^2, \\ 0, & D > \sigma^2. \end{cases}$$



- **Standard rate-distortion analysis** relies on the **stationarity** and the **ergodicity** (or the **information-stability**) of the source.

- However, DT WSACS processes are **nonstationary** and **nonergodic**.

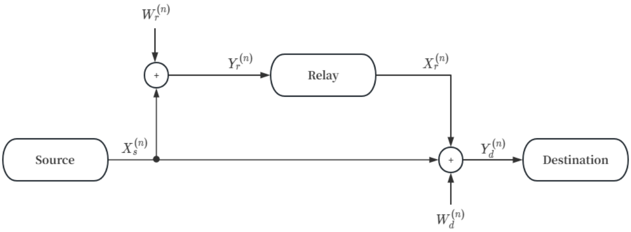
  $\Rightarrow$ Conventional information-theoretic arguments are **inapplicable**.

- Suitable frameworks for analyzing **information-unstable processes**:
  - **Asymptotically mean-stationary (AMS) processes** [Gray: 2011];
  - **Information-spectrum framework** [Han: 2010].

- We use the **information-spectrum framework** in our study.

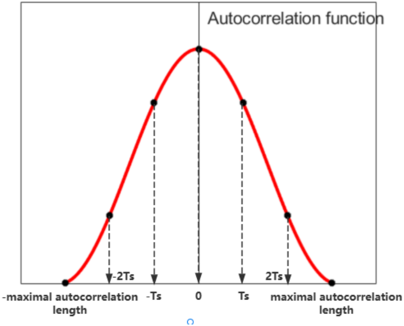
Outline

Compress-and-forward relay networks



- The (*Gaussian*) signal received at the **relay** is **sampled**,
  - before being **compressed** and **forwarded** to the destination.
- Practically, the **ratio** between the **actual sampling interval** at the relay and the **actual period of statistics** of the received signal is **irrational**, due to
  - **physical separation** of the **relay oscillator** and the **source oscillator**;
  - **inherent variability** of oscillators' frequencies (i.e., **jitter**).

- To reduce the **loss of information**, the **sampling interval** at the relay is typically *smaller* than the **maximal autocorrelation length** of the received signal.



Therefore, the sampled received signal at the relay is a **DT WSACS Gaussian process with memory.**

- [Kipnis et al.: *IEEE TIT* 2018] characterized the **distortion-rate function (DRF)** of **DT WSCS Gaussian processes**,
  - by transforming the **DT scalar WSCS process** into a **DT vector stationary process**.

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- The **capacity** of **additive DT WSACS Gaussian channels** was characterized using the **information-spectrum framework**,
  - in [Shlezinger et al.: *IEEE TCOM* 2020] for *memoryless noise*;
  - in [Dabora and Abakasanga: *IEEE TIT* 2023] for *noise with memory*.

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Our task: RDF characterization for compressing DT WSACS Gaussian processes with memory.

Outline

- Consider a sequence of real random variables (RVs) $\{X_i\}_{i=0}^{\infty}$:
  - The **limit superior in probability** is given as

$$\text{p-}\limsup_{i\to\infty} X_i \triangleq \inf\left\{\alpha \in \mathcal{R} \,\middle|\, \lim_{i\to\infty} \Pr\{X_i > \alpha\} = 0\right\};$$

  - Given a **common probability measure** P, if

$$\lim_{u\to\infty} \sup_{i\geq 0} \int_{|X_i|\geq u} |X_i| \mathrm{d}P = 0,$$

then the sequence is said to be **uniformly integrable**.

Consider a real DT **source process** $X[i]$, $i \in \mathcal{N}$, and a **reconstruction process** $\hat{X}[i]$:

- Denote a **block of l symbols** $\{X[i]\}_{i=0}^{l-1} \equiv \mathbf{X}^l$;
- Denote a **block of l reconstruction symbols** $\{\hat{X}[i]\}_{i=0}^{l-1} \equiv \hat{\mathbf{X}}^l$.
- Let $f_{\hat{\mathbf{X}}^l|\mathbf{X}^l}(\hat{\mathbf{x}}^l|\mathbf{x}^l)$ denote the **conditional probability density function (PDF)** of $\hat{\mathbf{X}}^l$ given $\mathbf{X}^l$.
- Let $f_{\hat{\mathbf{X}}^l}(\hat{\mathbf{x}}^l)$ denote the **PDF** of $\hat{\mathbf{X}}^l$.

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- Let $f_{\hat{\mathbf{X}}^l|\mathbf{X}^l}(\hat{\mathbf{x}}^l|\mathbf{x}^l)$ denote the **conditional PDF** of $\hat{\mathbf{X}}^l$ given $\mathbf{X}^l$.
- Let $f_{\hat{\mathbf{X}}^l}(\hat{\mathbf{x}}^l)$ denote the **PDF** of $\hat{\mathbf{X}}^l$.

- The **mutual information density rate** between $\mathbf{X}^l$ and $\hat{\mathbf{X}}^l$ is defined as

$$i(\mathbf{X}^l;\hat{\mathbf{X}}^l) \triangleq \frac{1}{l}\log\frac{f_{\hat{\mathbf{X}}^l|\mathbf{X}^l}(\hat{\mathbf{x}}^l|\mathbf{x}^l)}{f_{\hat{\mathbf{X}}^l}(\hat{\mathbf{x}}^l)}.$$

- The **spectral sup-mutual information rate** between $\mathbf{X}^\infty$ and $\hat{\mathbf{X}}^\infty$, $\bar{\mathrm{I}}(\mathbf{X}^\infty,\hat{\mathbf{X}}^\infty)$, is defined as

$$\bar{\mathrm{I}}(\mathbf{X}^\infty,\hat{\mathbf{X}}^\infty) \triangleq \text{p-}\limsup_{l\to\infty} i(\mathbf{X}^l;\hat{\mathbf{X}}^l).$$

- Consider a real **CT WSCS Gaussian** source process $X_c(t)$, s.t.
  - $\mathbb{E}\{X_c(t)\} = 0$.
  - AF $c_{x_c}(t,\lambda)$ is **periodic** in t with **period** T_c, **bounded** and **uniformly continuous** in both t and λ.
  - AF has **finite-memory** with maximal autocorrelation length λ_c,
    - i.e., $c_{x_c}(t,\lambda) \triangleq \mathbb{E}\{X_c(t) \cdot X_c(t+\lambda)\} = 0$, $\forall |\lambda| > \lambda_c$.

- Consider a real **CT WSCS Gaussian** source process $X_c(t)$, s.t.
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- $X_c(t)$ is *sampled* with **sampling interval** $T_s(\epsilon) \leq \lambda_c$ and **initial sampling phase** $\phi_s \in [0, T_c) \longrightarrow$ DT sampled process:

$$X_\epsilon^{\phi_s}[i] \triangleq X_c(T_s(\epsilon) \cdot i + \phi_s),$$

$T_s(\epsilon) \triangleq \frac{T_c}{p+\epsilon}$, where $p \in \mathcal{N}^+$ and $\epsilon \notin \mathcal{Q}$, $\epsilon \in (0,1)$ is the **sampling mismatch**.

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$T_s(\epsilon) \triangleq \frac{T_c}{p+\epsilon}$, where $p \in \mathcal{N}^+$ and $\epsilon \notin \mathcal{Q}$, $\epsilon \in (0, 1)$ is the **sampling mismatch**.

- Source sequences can be generated
  1. **continuously**, i.e., delay between consecutive sequences is **NOT** allowed;
  2. with **finite and bounded delay** between consecutive sequences.

  $\Rightarrow$ all sequences are **statistically independent**.

- For an **arbitrary** DT process $X[i]$, $i \in \mathcal{N}$, if for a **block of l symbols** $\{X[i]\}_{i=0}^{l-1} \equiv \mathbf{X}^l$, there exists a **deterministic reference word** $\{r_i\}_{i=0}^{l-1} \equiv \mathbf{r}^l$ which makes the **sequence of RVs** $\{\overline{d}(\mathbf{X}^l, \mathbf{r}^l)\}_{l=0}^{\infty}$ **uniformly integrable**.
  - Denote the **block of l reconstruction symbols** $\{\hat{X}_i\}_{i=0}^{l-1} \equiv \hat{\mathbf{X}}^l$ and the **joint cumulative distribution function (CDF)** of $\mathbf{X}^l$ and $\hat{\mathbf{X}}^l$ as $F_{\mathbf{X}^l, \hat{\mathbf{X}}^l}$.
  - Then, the **RDF** for compressing $X[i]$ using the **fixed-length coding** and the **average distortion criterion** is [Han: 2010]

$$R(D) = \inf_{\substack{F_{\mathbf{X}^l, \hat{\mathbf{X}}^l}: \\ \limsup_{l\to\infty} \mathbb{E}\{\overline{d}(\mathbf{X}^l, \hat{\mathbf{X}}^l)\} \leq D}} \bar{\mathrm{I}}(\mathbf{X}^l, \hat{\mathbf{X}}^l).$$

Outline

- **The RDF for compressing DT WSACS Gaussian processes with memory with continuous source sequence generation**:
  - Denote a **block of l symbols** $\{X_{\epsilon}^{\phi_s}[i]\}_{i=0}^{l-1} \equiv \mathbf{X}_{\epsilon,\phi_s}^{l}$, its **block of l reconstruction symbols** $\{\hat{X}_{\epsilon}^{\phi_s}[i]\}_{i=0}^{l-1} \equiv \hat{\mathbf{X}}_{\epsilon,\phi_s}^{l}$, and
  $$\mathbf{S}_{\epsilon,\phi_s}^{l} \triangleq \mathbf{X}_{\epsilon,\phi_s}^{l} - \hat{\mathbf{X}}_{\epsilon,\phi_s}^{l}.$$
  - Define the set
  $$\mathcal{C}_{\mathbf{S}_{\epsilon,\phi_s}^{l}} \triangleq \Big\{ \mathsf{C}_{\mathbf{S}_{\epsilon,\phi_s}^{l}} \in \mathcal{R}^{l\times l} \Big| \frac{1}{l}\operatorname{tr}\{\mathsf{C}_{\mathbf{S}_{\epsilon,\phi_s}^{l}}\} \leq D, \mathsf{C}_{\mathbf{S}_{\epsilon,\phi_s}^{l}} \succ 0, \mathsf{C}_{\mathbf{S}_{\epsilon,\phi_s}^{l}} = (\mathsf{C}_{\mathbf{S}_{\epsilon,\phi_s}^{l}})^T, \mathsf{C}_{\mathbf{X}_{\epsilon,\phi_s}^{l}} \succeq \mathsf{C}_{\mathbf{S}_{\epsilon,\phi_s}^{l}} \Big\}.$$

- Then the **RDF** $R_\epsilon(D)$ for compressing $X_\epsilon^{\phi_s}[i]$ is

$$R_\epsilon(D) = \frac{1}{T_c} \int_{\phi_s=0}^{T_c} R_\epsilon^{\phi_s}(D) \mathrm{d}\phi_s,$$

where

$$R_\epsilon^{\phi_s}(D) \triangleq \limsup_{l\to\infty} \min_{\mathrm{C}_{\mathbf{S}^l_{\epsilon,\phi_s}} \in \mathcal{C}_{\mathbf{S}^l_{\epsilon,\phi_s}}} \frac{1}{2l} \log\left(\frac{\det\left(\mathrm{C}_{\mathbf{X}^l_{\epsilon,\phi_s}}\right)}{\det\left(\mathrm{C}_{\mathbf{S}^l_{\epsilon,\phi_s}}\right)}\right),$$

in which $\mathbf{S}^l_{\epsilon,\phi_s}$ is a **Gaussian vector**.

- **Corollary (RDF with the fixed code rate)**
  - If the **code rate** has to be **fixed**, the **RDF** is

$$R_\epsilon(D) = \min_{\phi_s\in[0,T_c)} R_\epsilon^{\phi_s}(D).$$

Converse Part:

1. Consider a *specific* **sequence of lossy source codes** $(\tilde{M}(l,D), l)$ defined by the **encoder** $\tilde{f}_l(\cdot)$ and the **decoder** $\tilde{g}_l(\cdot)$, such that
$$\limsup_{l\to\infty} \mathbb{E}\{\overline{d}_{se}(\mathbf{X}^l_{\epsilon,\phi_s}, \hat{\tilde{\mathbf{X}}}^l_{\epsilon,\phi_s})\} \leq D,$$
where $\hat{\tilde{\mathbf{X}}}^l_{\epsilon,\phi_s} = \tilde{g}_l\big(\tilde{f}_l(\mathbf{X}^l_{\epsilon,\phi_s})\big)$.

2. By [Kostina2013], we obtain
$$\log \tilde{M}(l,D) \geq \inf_{\substack{f(\hat{\mathbf{X}}^l_{\epsilon,\phi_s}|\mathbf{X}^l_{\epsilon,\phi_s}):\\ \mathbb{E}\{\overline{d}_{se}(\mathbf{X}^l_{\epsilon,\phi_s},\hat{\mathbf{X}}^l_{\epsilon,\phi_s})\}\leq\mathbb{E}\{\overline{d}_{se}(\mathbf{X}^l_{\epsilon,\phi_s},\hat{\tilde{\mathbf{X}}}^l_{\epsilon,\phi_s})\}}} I(\mathbf{X}^l_{\epsilon,\phi_s};\hat{\mathbf{X}}^l_{\epsilon,\phi_s}).$$

3. By **asymptotic properties** of the **limit superior** and the condition $\limsup_{l\to\infty} \mathbb{E}\{\overline{d}_{se}(\mathbf{X}^l_{\epsilon,\phi_s}, \hat{\hat{\mathbf{X}}}^l_{\epsilon,\phi_s})\} \leq D$, we obtain

$$R^{\phi_s}_{\epsilon}(D) \geq \limsup_{l\to\infty} \inf_{\substack{f(\hat{\mathbf{X}}^l_{\epsilon,\phi_s}|\mathbf{X}^l_{\epsilon,\phi_s}): \\ \mathbb{E}\{\overline{d}_{se}(\mathbf{X}^l_{\epsilon,\phi_s},\hat{\mathbf{X}}^l_{\epsilon,\phi_s})\}\leq D}} \frac{1}{l} I(\mathbf{X}^l_{\epsilon,\phi_s}; \hat{\mathbf{X}}^l_{\epsilon,\phi_s}).$$

4. Minimizing $\frac{1}{l} I(\mathbf{X}^l_{\epsilon,\phi_s}; \hat{\mathbf{X}}^l_{\epsilon,\phi_s})$, the **optimal** $\mathbf{S}^l_{\epsilon,\phi_s}$ is found to be **Gaussian**, therefore

$$R^{\phi_s}_{\epsilon}(D) \geq \limsup_{l\to\infty} \min_{\mathsf{C}_{\mathbf{S}^l_{\epsilon,\phi_s}} \in \mathcal{C}_{\mathbf{S}^l_{\epsilon,\phi_s}}} \frac{1}{2l} \log\left(\frac{\det\left(\mathsf{C}_{\mathbf{X}^l_{\epsilon,\phi_s}}\right)}{\det\left(\mathsf{C}_{\mathbf{S}^l_{\epsilon,\phi_s}}\right)}\right).$$

Achievability Part:

1. Denote the **optimal block of l reconstruction symbols** as $\{\hat{X}^{l,*}_{\epsilon,\phi_s}\}_{i=0}^{l-1} \equiv \hat{\mathbf{X}}^{l,*}_{\epsilon,\phi_s}$ and the **mutual information density rate** between $\mathbf{X}^{l}_{\epsilon,\phi_s}$ and $\hat{\mathbf{X}}^{l,*}_{\epsilon,\phi_s}$ as $\underline{Z_l(F^{\phi_s}_{X_\epsilon,\hat{X}^*_\epsilon})}$.

   The **mutual information density** $V^{\phi_s}_{\epsilon,l}$ between $\mathbf{X}^{l}_{\epsilon,\phi_s}$ and $\hat{\mathbf{X}}^{l,*}_{\epsilon,\phi_s}$ is

$$V^{\phi_s}_{\epsilon,l} = l \cdot \underline{Z_l(F^{\phi_s}_{X_\epsilon,\hat{X}^*_\epsilon})} \overset{\text{dist.}}{=} \underbrace{\frac{1}{2}\log\left(\frac{\det\left(\mathsf{C}_{\mathbf{X}^{l}_{\epsilon,\phi_s}}\right)}{\det\left(\mathsf{C}_{\mathbf{S}^{l,*}_{\epsilon,\phi_s}}\right)}\right)}_{\text{a constant}} + \underbrace{\frac{1}{2}\log(e)\cdot\underline{\tilde{V}^{l}_{\epsilon,\phi_s}}}_{\text{a RV}}.$$

2. Rewrite $\underline{\tilde{V}^{l}_{\epsilon,\phi_s}} \overset{\text{dist.}}{=} \sum_{i=0}^{2l-\tilde{l}_{\epsilon,l}-1} \lambda^{\phi_s}_{\epsilon,i} \cdot \underline{\underline{(\tilde{\gamma}^{\phi_s}_{\epsilon,i})^2}}$, where $2l-\tilde{l}_{\epsilon,l}-1 \in \mathcal{N}$, $\lambda^{\phi_s}_{\epsilon,i} \in \mathcal{R}$ and $\underline{\underline{(\tilde{\gamma}^{\phi_s}_{\epsilon,i})^2}}$ are **mutually independent central chi-square RVs** with **single degrees of freedom**.

3. Calculate $\mathbb{E}\{\underline{\tilde{V}^l_{\epsilon,\phi_s}}\}$ and $\text{Var}\{\underline{\tilde{V}^l_{\epsilon,\phi_s}}\}$, then $\mathbb{E}\{\underline{Z_l(F^{\phi_s}_{X_\epsilon,\hat{X}^*_\epsilon})}\}$ and $\text{Var}\{\underline{Z_l(F^{\phi_s}_{X_\epsilon,\hat{X}^*_\epsilon})}\}$ are obtained.

4. By the **Chebyshev inequality** and the **limit-superior in probability**, we obtain

$$R^{\phi_s}_\epsilon(D) \leq \limsup_{l\to\infty} \min_{\mathsf{C}_{\mathbf{S}^l_{\epsilon,\phi_s}} \in \mathcal{C}_{\mathbf{S}^l_{\epsilon,\phi_s}}} \frac{1}{2l} \log \left(\frac{\det\left(\mathsf{C}_{\mathbf{X}^l_{\epsilon,\phi_s}}\right)}{\det\left(\mathsf{C}_{\mathbf{S}^l_{\epsilon,\phi_s}}\right)} \right).$$

- Combining the **converse** and the **achievability**, we obtain

$$R_{\epsilon}^{\phi_s}(D) = \limsup_{l\to\infty} \min_{\mathsf{C}_{\mathbf{s}^l_{\epsilon,\phi_s}} \in \mathcal{C}_{\mathbf{s}^l_{\epsilon,\phi_s}}} \frac{1}{2l} \log\left(\frac{\det\left(\mathsf{C}_{\mathbf{X}^l_{\epsilon,\phi_s}}\right)}{\det\left(\mathsf{C}_{\mathbf{S}^l_{\epsilon,\phi_s}}\right)} \right).$$

- Considering the **continuousness** of the source sequence generation and the **equidistribution theorem** [Kuipers and Niederreiter: 1974], [Coppel: 2009],
  - as the number of source sequences goes to **infinity**, the **sampling phase** of each source sequence becomes *asymptotically* **uniformly distributed** over $[0, T_c)$.
  - Therefore, the **RDF** for compressing $X_{\epsilon}^{\phi_s}[i]$ is

$$R_{\epsilon}(D) = \frac{1}{T_c}\int_{\phi_s=0}^{T_c} R_{\epsilon}^{\phi_s}(D)\mathrm{d}\phi_s.$$

- **Remark 1**:
  - Intuitively, as $\epsilon \notin \mathcal{Q}$, the AF of $X_{\epsilon}^{\phi_s}[i]$ **never repeats itself** and *asymptotically* **exhibits all values** of $c_{X_c}(t, \lambda)$ as the **blocklength** l goes to **infinity**.
  - Therefore, we conjecture that $R_{\epsilon}^{\phi_s}(D)$ should **NOT** be affected by the **initial sampling phase** $\phi_s \in [0, T_c)$, therefore

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- **Remark 2**:
  - When the sampling is **synchronous** ($\epsilon \in \mathcal{Q}$), the **RDF** is **affected** by the **initial sampling phase** $\phi_s \in [0, T_c)$, namely

$$R_{\epsilon}^{sync}(D) = R_{\epsilon}^{\phi_s}(D),$$

    which goes back to the result in [Kipnis et al.: *IEEE TIT* 2018].
  - In this case, if the **RDF** has to be **fixed** and **minimized**, we have

$$R_{\epsilon}^{sync}(D) = \min_{\phi_s \in [0, T_c)} R_{\epsilon}^{\phi_s}(D).$$

Outline

Auxiliary Theorem: The **RDF** for **DT WSCS Processes**:

1. Let the **source process** $X_c(t)$ be sampled with **interval** $T_s(\epsilon_{\mathbf{n}}) \triangleq \frac{T_c}{p+\epsilon_{\mathbf{n}}}$, $\epsilon_{\mathbf{n}} \triangleq \frac{\lfloor n \cdot \epsilon \rfloor}{n} \in \mathcal{Q}$, $n \in \mathcal{N}^+$, and with an **initial sampling phase** $\phi_{\mathbf{s}} \in [0, T_c)$:

$$X_{\epsilon_{\mathbf{n}}}^{\phi_s}[i] \triangleq X_c\big(i \cdot T_s(\epsilon_{\mathbf{n}}) + \phi_{\mathbf{s}}\big).$$

$X_{\epsilon_{\mathbf{n}}}^{\phi_s}[i]$ is a **DT WSCS process** with

- maximal autocorrelation length $\tau_c \triangleq \left\lceil \frac{(p+1)\cdot\lambda_c}{T_c} \right\rceil \geq \left\lceil \frac{(p+\epsilon_{\mathbf{n}})\cdot\lambda_c}{T_c} \right\rceil$;
- period of statistics $p_n \triangleq p \cdot n + \lfloor n \cdot \epsilon \rfloor$.

2. Transform $X^{\phi_s}_{\epsilon_{\mathbf{n}}}[i]$ into a **p_n-dimensional stationary process**:

$$\left(\mathbf{X}^{p_n}_{\epsilon_{\mathbf{n}},\phi_s}[i]\right)_m \triangleq X^{\phi_s}_{\epsilon_{\mathbf{n}}}[i \cdot p_n + m], \quad m = 0, 1, \ldots, p_n - 1.$$

- **Autocorrelation matrix**

$$\mathsf{C}_{\mathbf{X}^{p_n}_{\epsilon_{\mathbf{n}},\phi_s}}[\Delta] \triangleq \mathbb{E}\left\{\mathbf{X}^{p_n}_{\epsilon_{\mathbf{n}},\phi_s}[i] \cdot \left(\mathbf{X}^{p_n}_{\epsilon_{\mathbf{n}},\phi_s}[i+\Delta]\right)^T\right\}, \quad \Delta \in \mathcal{Z};$$

- **Power spectral density (PSD) matrix**

$$\mathsf{S}_{\mathbf{X}^{p_n}_{\epsilon_{\mathbf{n}},\phi_s}}(f) \triangleq \sum_{\Delta \in \mathcal{Z}} \mathsf{C}_{\mathbf{X}^{p_n}_{\epsilon_{\mathbf{n}},\phi_s}}[\Delta] \cdot e^{-j2\pi f}, \quad -\frac{1}{2} \leq f \leq \frac{1}{2}.$$

- Let $\underline{\lambda^{p_n}_{\epsilon_{\mathbf{n}},\phi_s,m}(f)}$, $0 \leq m \leq p_n - 1$, denote the **eigenvalues** of $\mathsf{S}_{\mathbf{X}^{p_n}_{\epsilon_{\mathbf{n}},\phi_s}}(f)$.

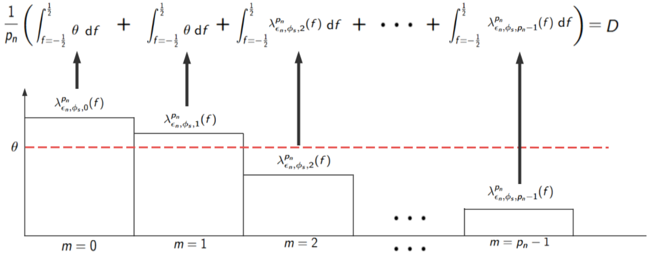
3. Given the **distortion constraint** D, the **RDF** for compressing $X_{\epsilon_n}^{\phi_s}[i]$ is [Kipnis et al.: *IEEE TIT* 2018]

$$R_{\epsilon_n}^{\phi_s}(D) = \frac{1}{2p_n} \sum_{m=0}^{p_n-1} \int_{f=-\frac{1}{2}}^{\frac{1}{2}} \left(\log \left(\frac{\lambda_{\epsilon_n,\phi_s,m}^{p_n}(f)}{\theta} \right) \right)^+ \mathrm{d}f,$$

where θ is selected s.t.

$$D = \frac{1}{p_n} \sum_{m=0}^{p_n-1} \int_{f=-\frac{1}{2}}^{\frac{1}{2}} \min \left\{ \lambda_{\epsilon_n,\phi_s,m}^{p_n}(f), \theta \right\} \mathrm{d}f.$$

- **Reverse waterfilling**



RDF characterization for $X_{\epsilon}^{\phi_s}[i]$:

1. Define $R_{\epsilon_n}(D) \triangleq \min_{\phi_s \in [0, T_c)} R_{\epsilon_n}^{\phi_s}(D)$.
2. If the **AF** of $X_c(t)$ satisfies
$$\min_{0 \le t < T_c} \left\{ c_{X_c}(t, 0) - 2 \cdot \tau_c \cdot \max_{|\lambda| > \frac{T_c}{p+1}} \left\{ \left| c_{X_c}(t, \lambda) \right| \right\} \right\} \ge \gamma_c > 0,$$
3. for the **distortion constraint** $D \le \gamma_c$,
4. when a **finite and bounded delay** between processed sequences up to $\tau_c \cdot T_s(\epsilon) + T_c$ in CT is allowed,

the **RDF** for compressing $X_{\epsilon}^{\phi_s}[i]$ is given as
$$R_{\epsilon}(D) = \limsup_{n \to \infty} R_{\epsilon_n}(D).$$

- **Gaussian asymptotically wide-sense stationary (AWSS) vector processes** [Gutiérrez-Gutiérrez et al.: *Entropy* 2018]
  - Such processes have **a constant mean** and their $n \times n$ **autocorrelation matrices** are **asymptotically block Toeplitz**.
  - **DT WSACS processes** *CANNOT* be transformed into **AWSS vector processes**.

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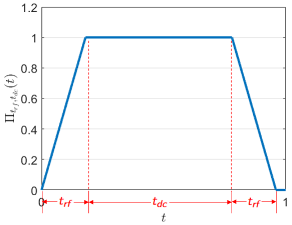
  - **DT WSACS processes** *CANNOT* be transformed into **AWSS vector processes**.
- **AMS processes** [Gray: 2011]
  - **DT WSACS processes** are **AMS** [Gardner: *J. Sound Vib.* 2023].
  - For an **AMS process** with **reference letter**, its **DRF** is *equivalent* to the **DRF** of its **associated stationary source**.
  - In our scenario, we were *NOT* able to evaluate the AMS **stationary mean**, i.e., **probability measure** of a **Gaussian mixture** as the number of summands goes to **infinity**.

Setup of CT AF:

1. Consider a **periodic pulse function** $\Pi_{t_{rf}, t_{dc}}(t)$, $t_{rf} = 0.01$, $t_{dc} \in [0, 0.98]$ and **period** 1:



2. Set **period of AF** of $X_c(t)$ as $\mathbf{T_c} = 5$ µsec.

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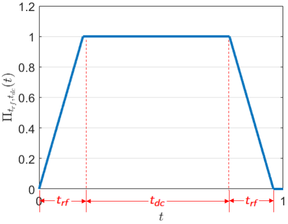

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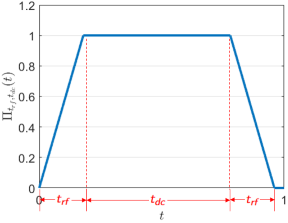

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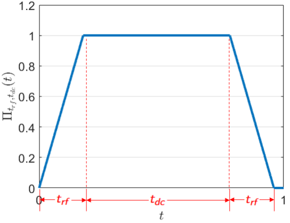

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$$c_{X_c}(t, \lambda) \triangleq \mathbf{1}\left(\lambda \in [0, \lambda_c]\right) \cdot e^{-\lambda \cdot 10^{6.1}} \cdot c_{X_c}(t, 0).$$

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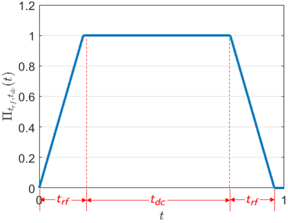

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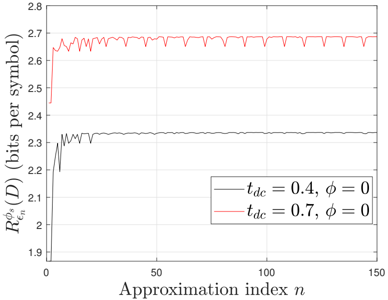
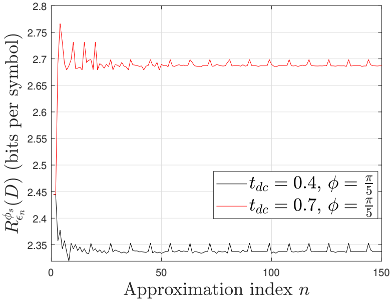
6. For $\lambda < 0$, $c_{X_c}(t, \lambda) = c_{X_c}(t + \lambda, -\lambda)$.

The **RDF** $R_{\epsilon_n}(D)$ w.r.t. **approximation index** n for $\epsilon = \frac{\pi}{7}$:





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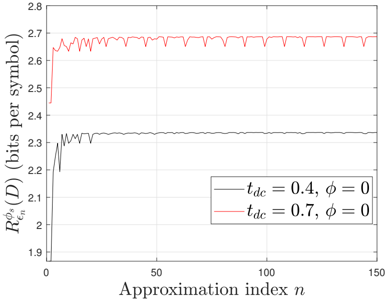

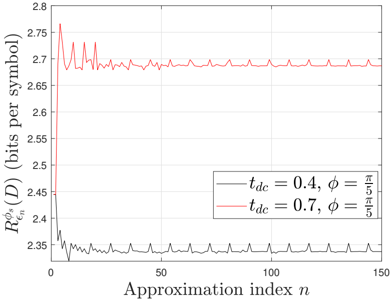

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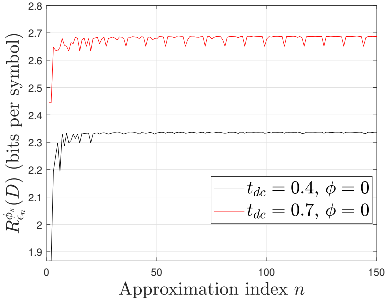

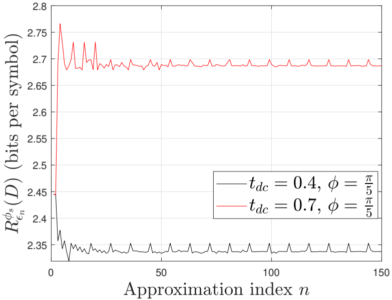

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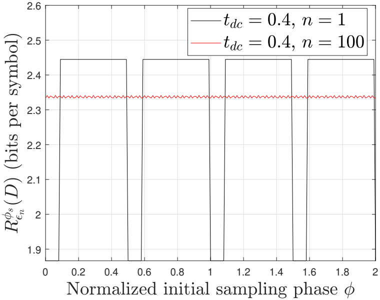
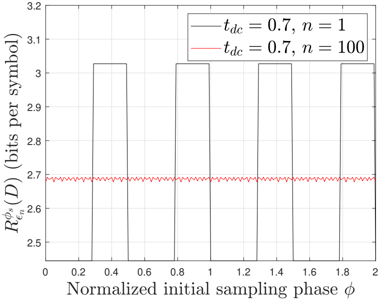
  - Larger n $\longrightarrow$ larger **period of statistics** p_n $\longrightarrow$ shifting of sampling instances **negligible**.
- $R_{\epsilon_n}(D)$ does *NOT* converge.
  - **Asynchronism** of sampling $\longrightarrow$ **nonstationarity**.

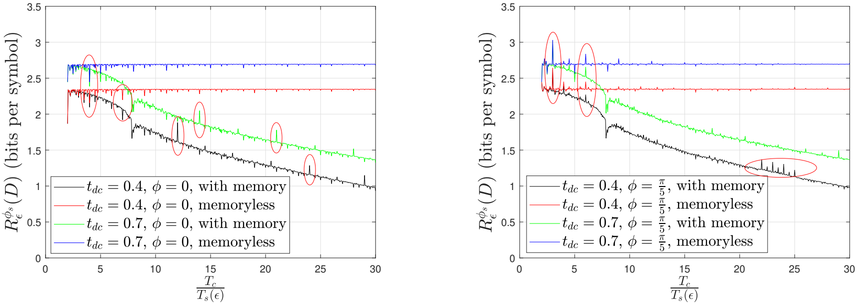
The **RDF** w.r.t. **normalized initial sampling phase** ϕ for $\epsilon = \frac{\pi}{7}$:





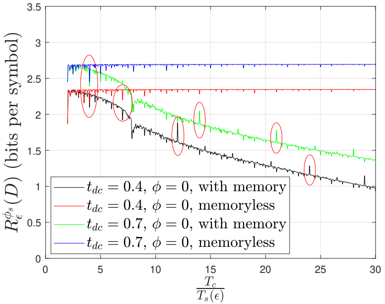
- When n is **large enough**, $R_{\epsilon_n}(D)$ varies **very little** w.r.t. **normalized initial sampling phase**
  - Larger $n \longrightarrow$ more **CT cycles** covered in **one DT period** $\longrightarrow$ impact of **shifting** of a few samples on the rate **negligible**.

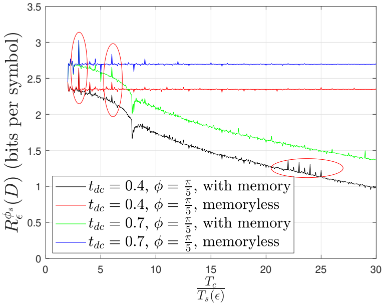
The **RDF** w.r.t. **sampling frequency**:



- RDF for **processes with memory** is generally *smaller* than RDF for **memoryless processes**.

The **RDF** w.r.t. **sampling frequency**:

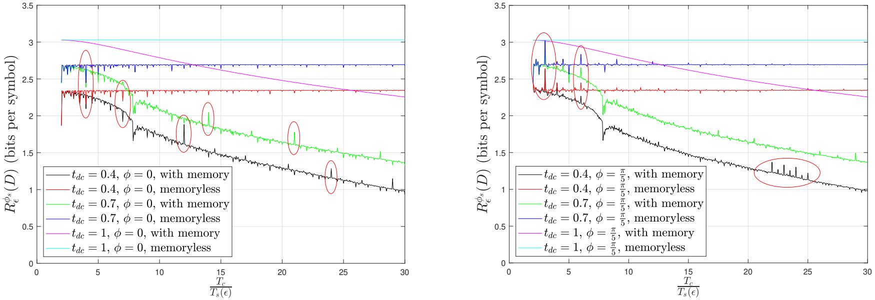




- RDF for **processes with memory** is generally *smaller* than RDF for **memoryless processes**.
- **Small sampling rate variations** could lead to **large compression rate variations**.

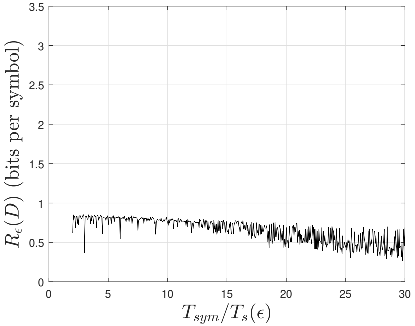
Practical insight: In design of communications systems, assume **asynchronous sampling**.

RDF for sampled **WSCS** processes vs **RDF** for sampled **stationary** processes



- **RDF curve** for the **stationary** sampled process is **smooth**.
- **RDF curve** for the **memoryless stationary** sampled process is **flat**.
  - Compression of an **i.i.d.** source.

The **RDF** of **real passband OFDM signal** w.r.t. **sampling frequency**:



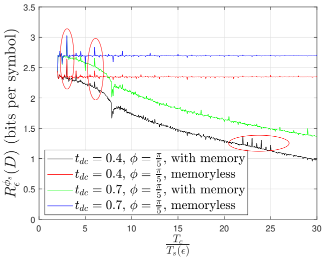
- **Real passband OFDM signal** is a **CT WSCS** process
  - with the **period of statistics** equaling to its **symbol period**.
- RDF exhibits **fluctuations**, demonstrating the **practical insight**.

Practical insight: In design of communications systems, assume **asynchronous sampling**.

Outline

- We defined the problem of **RDF characterization** for **DT WSACS Gaussian processes with memory**,
  - without delay between consecutive sequences.
  - with **finite and bounded processing delay** between consecutive sequences.
- This problem arises in practical sampling scenarios:
  - e.g., **compress-and-forward relay networks**.
- We presented the **RDF results** for compressing **DT WSACS Gaussian processes with memory** for two source sequence generation schemes.
- We discussed **additional** frameworks for RDF analysis and notions implied from **numerical evaluations**.
  - **Gaussian AWSS vector processes** and **AMS processes** are *NOT* useful for RDF analysis in our problem.

- **Important insight**: Assume **asynchronous sampling** in **practical communications systems design**:



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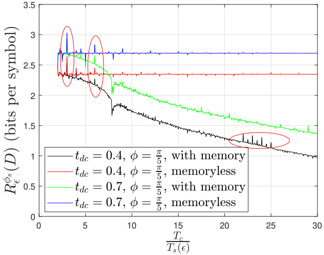

- **Future research directions**:
  - **Compress-and-forward relaying**: Clock synchronization mismatch **between the relay and the source** and **between the relay and the destination**.

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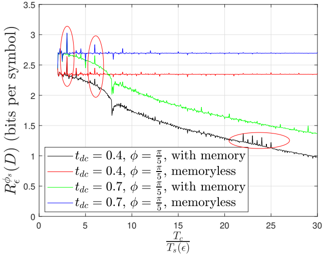

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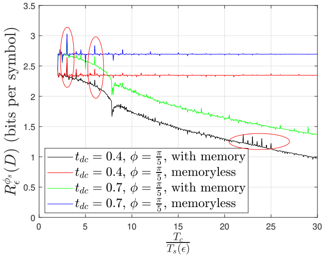
Summary and Future Works

- **Important insight**: Assume **asynchronous sampling** in **practical communications systems design**:



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  - **Wyner-Ziv model** in multiuser compression, where the **sampling clock** of the **side information** is *NOT* synchronized with that of the **source**.
  - **Secrecy capacity** in the presence of an **eavesdropper**
    - Multiple settings depending on which receivers (i.e., destination or eavesdropper) observes interference.